Construction of finite W-algebras

1) Slodowy slices
2) Construction via Hamiltonian reduction.
3) Construction via quantum slices.
4) What's next?

0) Notation

Base field: $\mathbb{C}$

G s/simple alg. group, $\mathfrak{g}=\mathrm{Lie}(G)$.

$(\cdot,\cdot)=$ Killing form on $\mathfrak{g} \rightsquigarrow \mathfrak{g}\simeq\mathfrak{g}^*$

$\mathbb{O}\subset\mathfrak{g}$ nilpotent G-orbit

nilpotent matrices for classical G

$e\in\mathbb{O}$ can be included into $\mathfrak{sl}_2$-triple (e,h,f) by Jacobson-Morozov theorem. By Kostant, (e,h,f) is defined uniquely up to $Z_G(e)$-action.

1) Slodowy slices

Goals: 1) Define a transverse slice S to $\mathbb{O}$ in $\mathfrak{g}$, an affine space.

2) Equip $\mathbb{C}[S]$ with Poisson bracket.

3) Quantize $\mathbb{C}[S]$ (to finite W-algebra). We will offer two construction of the bracket on $\mathbb{C}[S]$ (via Hamiltonian reduction & slice) & both will admit direct quantization.

1.1) Construction of S & contracting action

Set $S := e + \mathfrak{z}_{\mathfrak{g}}(f)$, affine subspace of $\mathfrak{g}$

centralizer, ker(ad f).

Exercise 1: Show that $\mathfrak{g} = T_e S \oplus T_e \mathbb{O}$ (hint: observe that $T_e \mathbb{O} = \operatorname{im} \operatorname{ad}(e)$ & use rep. theory of $\mathfrak{sl}_2$)

An important tool to study S is a $\mathbb{C}^\times$-action. Let $\gamma: \mathbb{C}^\times \longrightarrow G$ be the composition of:

- $\mathbb{C}^\times \longrightarrow SL_2$, $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$
- & $SL_2 \rightarrow G$ induced by the triple (e,h,f)

Exercise 2: Consider the action of $\mathbb{C}^\times$ on $\mathfrak{g}$ given by $t \cdot x = t^{-2} Ad(\gamma(t))x$ $(t \in \mathbb{C}^\times, x \in \mathfrak{g})$, the modified action.

Then this action fixes e and preserves S. Moreover,

(1) $$\lim_{t \to \infty} t \cdot s = e \quad \forall s \in S$$

(hint: the e-values of $ad(h)$ on $\mathfrak{z}_{\mathfrak{g}}(f) = \ker(ad(f))$ are in $\mathbb{Z}_{\leq 0}$)

Note that we have the following algebraic reformulation:

$\mathbb{C}^\times \curvearrowright S \rightsquigarrow \mathbb{Z}$-grading on $\mathbb{C}[S]$, $\mathbb{C}[S]_i = \{f \in \mathbb{C}[S] \mid t.f = t^i f\}$

(1) $\Leftrightarrow$ the grading is positive: $\mathbb{C}[S]_i = 0$ for $i < 0$, $\mathbb{C}[S]_0 = \mathbb{C}$.

Corollary: 1) $S \cap \overline{\mathbb{O}} = \{e\}$

2) For a G-orbit $\mathbb{O}' \subset \mathfrak{g}$: $S \cap \mathbb{O}' \neq \emptyset \Leftrightarrow \overline{\mathbb{C}^\times \mathbb{O}'} \supset \mathbb{O}$.

3) If $x \in S \cap \mathbb{O}'$, then $T_x S + T_x \mathbb{O}' = \mathfrak{g}$

Special case (Kostant): Let $\mathbb{O}$ be principal ($\overline{\mathbb{O}}$ = nilpotent cone i.e. the locus of nilpotent elements); $\pi: \mathfrak{g} \to \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G$, quotient morphism. Then $\pi|_S$ is iso.

2) Construction via Hamiltonian reduction.

2.1) Poisson structure on $\mathbb{C}[S]$.

Suppose $N \subset G$ is a connected subgroup, $\mathfrak{n} = Lie(N)$ & $\psi \in (\mathfrak{n}^*)^N$. Consider $\mu: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$, where the 2nd map is the restriction map, $\alpha \mapsto \alpha|_{\mathfrak{n}}$, then μ is a "moment map." It is N-equivariant so $N \curvearrowright \mu^{-1}(\psi) \rightsquigarrow$ algebra $\mathbb{C}[\mu^{-1}(\psi)]^N$. Consider $I = Span_{S(\mathfrak{g})}(x - \psi(x) \mid x \in \mathfrak{n}) \subset S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ so that $\mathbb{C}[\mu^{-1}(\psi)] = S(\mathfrak{g})/I$.

Note that $S(\mathfrak{g})$ carries a Poisson bracket $\{\cdot,\cdot\}$ extending $[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Exercise 1 (classical Hamiltonian reduction): $\{a+I, b+I\} := \{a,b\} + I$ gives a well-defined Poisson bracket on $\mathbb{C}[\mu^{-1}(\psi)]^N$.

We can find N & ψ s.t. $\mathbb{C}[S]$ is identified w. $\mathbb{C}[\mu^{-1}(\psi)]^N$. Namely, let $\mathfrak{g}(i) := \{x \in \mathfrak{g} \mid [h,x] = ix\}, i \in \mathbb{Z}$.

Exercise 2: $(x,y) \mapsto (e, [x,y])$ is a symplectic form on $\mathfrak{g}(-1)$.

Pick a Lagrangian subspace $\ell \subset \mathfrak{g}(-1)$ and set

$$\mathfrak{k} = \ell \oplus \bigoplus_{i<-2} \mathfrak{g}(i).$$

This is an ad-nilpotent subalgebra so we can set $N := \exp(\mathfrak{k}) \subset G$.

Exercise 3: 1) $\psi := (e, \cdot)|_{\mathfrak{k}}$ is N-invariant

2) $S \subset \mu^{-1}(\psi)$

Fact (Gan-Ginzburg): The morphism $N \times S \to \mu^{-1}(\psi)$ $(n,s) \mapsto Ad(n)s$ is iso.

This gives $\mathbb{C}[S] \xrightarrow{\sim} \mathbb{C}[\mu^{-1}(\psi)]^N$, hence $\{\cdot,\cdot\}$ on $\mathbb{C}[S]$. One can recover the grading on $\mathbb{C}[\mu^{-1}(\psi)]^N$ as follows. Define a new grading of $S(\mathfrak{g})$ by $\deg \mathfrak{g}(i) = i+2$. Note that $I \subset S(\mathfrak{g})$ is a homogeneous ideal, yielding a grading on $\mathbb{C}[\mu^{-1}(\psi)]^N$.

Exercise 4: $\mathbb{C}[S] \xrightarrow{\sim} \mathbb{C}[\mu^{-1}(\psi)]^N$ is graded and $\deg\{\cdot,\cdot\} = -2$.

Special case: Suppose $\mathbb{O}$ is principal. We can choose $e = \sum_{i=1}^{r} e_i$, the sum of simple root vectors & $h = 2\rho^\vee$, the sum of all coroots. The $\mathfrak{n} = \sum_{\alpha<0} \mathfrak{g}_\alpha$ the opposite maximal nilpotent subalgebra. The isomorphism $\mathbb{C}[S] \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G$ is Poisson showing that the bracket on $\mathbb{C}[S]$ is 0.

2.2) Quantization.

Our goal here is to construct a filtered quantization $\mathcal{W}$ of a graded Poisson algebra $\mathbb{C}[S]$, i.e. an associative algebra $\mathcal{W}$ w. algebra filtration $\mathcal{W} = \bigcup_{i\geq 0} \mathcal{W}_{\leq i}$ satisfying $[\mathcal{W}_{\leq i}, \mathcal{W}_{\leq j}] \subset \mathcal{W}_{\leq i+j-2}$ $\forall\, i,j$ s.t. $\operatorname{gr}\mathcal{W} \xrightarrow{\sim} \mathbb{C}[S]$ as graded Poisson algebras.

We start with the construction via quantum Hamiltonian reduction, which is very close to the original construction of Premet, interpreted in this way by Gan &

Ginzburg.

Set $\mathcal{W} = [\mathcal{U}(\mathfrak{g}) / \underbrace{\mathcal{U}(\mathfrak{g})\{x - \langle\psi, x\rangle \mid x \in \mathfrak{n}\}}_{\overset{\shortparallel}{J}}]^N$

This has a well-defined product: $(a+J)(b+J) := ab + J$, & a filtration induced from the modified filtration on $\mathcal{U}(\mathfrak{g})$ (w. $\deg \mathfrak{g}(i) = i+2$). With this filtration, we get

$$\operatorname{gr} \mathcal{W} \hookrightarrow \mathbb{C}[S]$$

That this is an isomorphism can be deduced from Fact in Section 2.1.

Special case: Note that the center $\mathcal{Z}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^G$ admits a natural homomorphism to $\mathcal{W}$. It is filtered where we consider the restriction of the doubled PBW filtration (where $\deg \mathfrak{g} = 2$) on $\mathcal{Z}(\mathfrak{g})$. The associated graded of this homomorphism is $\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[S]$, the pullback under $S \to \operatorname{Spec} \mathbb{C}[\mathfrak{g}]^G$. It follows that when $\mathbb{O}$ is principal, we have $\mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{W}$, a result going back to Kostant.

3) Construction via quantum slices.

3.1) Slice construction of $\{\cdot,\cdot\}$

Let $\mathfrak{m} \subset \mathbb{C}[\mathfrak{g}]$ be the maximal ideal of e. Consider the completion $\mathbb{C}[\mathfrak{g}]^{\wedge e} := \varprojlim_n \mathbb{C}[\mathfrak{g}]/\mathfrak{m}^n$, the algebra of formal powers series. Similarly we can consider completions $\mathbb{C}[\bar{\mathbb{O}}]^{\wedge e} \xrightarrow{\sim} \mathbb{C}[[T_e\mathbb{O}]]$ & $\mathbb{C}[S]^{\wedge e}$. Since S intersects $\mathbb{O}$ transversally at a single point, we have noncanonical isomorphism

completed tensor product

(2) $\mathbb{C}[[T_e\mathbb{O}]] \hat{\otimes}_{\mathbb{C}} \mathbb{C}[S]^{\wedge e} \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^{\wedge e}$

Note that both $\mathbb{C}[[T_e\mathbb{O}]]$, $\mathbb{C}[\mathfrak{g}]^{\wedge e}$ carry natural Poisson structures ($T_e\mathbb{O}$ is a symplectic vector space).

Proposition: There is a deg -2 (for the modified $\mathbb{C}^\times$-action) Poisson bracket on $\mathbb{C}[S]$ & a Poisson $\mathbb{C}^\times$-equivariant choice of isomorphism (2).

Sketch of proof: details are an extended exercise.

Let $V := (T_e\mathbb{O})^*$, symplectic w. form, say ω, of deg 2 for the $\mathbb{C}^\times$-action induced from the modified $\mathbb{C}^\times \curvearrowright \mathfrak{g}$.

1) We need $\mathbb{C}^\times$-equivariant map $\iota: V \to \mathbb{C}[\mathfrak{g}]^{\wedge e}$ s.t.

(*) $\{\iota(u), \iota(v)\} = \omega(u,v)$

(whose existence would follow from Proposition).

The decomposition $T_e\mathfrak{g} \xrightarrow{\sim} T_e\mathbb{O} \oplus T_e S$ yields $\iota_2: V \to \mathbb{C}[\mathfrak{g}]^{\wedge e}$ s.t. (*) holds mod $\mathfrak{m}^2$. We can $\mathbb{C}^\times$-equivariantly lift ι_2 to ι_3 (w. $\iota_2 = \iota_3$ mod $\mathfrak{m}^2$) s.t. (*) holds mod $\mathfrak{m}^3$, etc. Take $\iota := \varprojlim_j \iota_j$.

2) Poisson centralizer $\{f \in \mathbb{C}[\mathfrak{g}]^{\wedge e} \mid \{\iota(v), f\} = 0 \ \forall v \in V\}$ is $\mathbb{C}^\times$-equivariantly identified w. $\mathbb{C}[S]^{\wedge e}$ hence acquires a Poisson brecket that as all Poisson brackets involved has deg -2 w.r.t. the $\mathbb{C}^\times$-action. So it restricts to the subalgebra $\mathbb{C}[S]^{\wedge e}_{\mathbb{C}^\times\text{-fin}} = \{f \in \mathbb{C}[S]^{\wedge e} \mid f$ lies in a fin. dim. $\mathbb{C}^\times$-submodule$\}$. Since the $\mathbb{C}^\times$-action on S is contracting, this subalgebra coincides w. $\mathbb{C}[S]$ $\square$

Remarks: 1) Any choices of ι are conjugate by Hamiltonian automorphisms of $\mathbb{C}[\mathfrak{g}]^{\wedge e}$. This shows that the bracket on $\mathbb{C}[S]$ is well-defined up to a graded automorphism.

2) This construction gives a bracket isomorphic to one in Sec 1.2 but this is not obvious.

3.2) Quantum slice construction

Now I explain the construction that was found in my work from late 2000's, it quantizes the construction in Sec 3.1.

Equip $\mathcal{U}(\mathfrak{g})$ with the doubled PBW filtration and consider the Rees algebra $\mathcal{U}_\hbar(\mathfrak{g})$, explicitly,

$$\mathcal{U}_\hbar(\mathfrak{g}) = T(\mathfrak{g})[\hbar]/(x\otimes y - y\otimes x - [x,y]\hbar^2 \mid x,y\in\mathfrak{g})$$

It's graded w. $\deg \mathfrak{g} = 2$, $\deg \hbar = 1$.

In particular, $\mathcal{U}_\hbar(\mathfrak{g})/(\hbar) = S(\mathfrak{g})\ (\simeq \mathbb{C}[\mathfrak{g}])$. Let $\tilde{m}$ denote the preimage in $\mathcal{U}_\hbar(\mathfrak{g})$ of the maximal ideal $m \subset \mathbb{C}[\mathfrak{g}]$ of e. Then we can form the completion:

$$\mathcal{U}_\hbar(\mathfrak{g})^{\wedge e} := \varprojlim_n \mathcal{U}_\hbar(\mathfrak{g})/\tilde{m}^n.$$

Note that $\mathcal{U}_\hbar(\mathfrak{g})^{\wedge e}/(\hbar) = \mathbb{C}[\mathfrak{g}]^{\wedge e}$

Fact: $\hbar$ is not a zero divisor in $\mathcal{U}_\hbar(\mathfrak{g})^{\wedge e}$

So $\mathcal{U}_\hbar(\mathfrak{g})^{\wedge e}$ is a deformation quantization of $\mathbb{C}[\mathfrak{g}]^{\wedge e}$. The algebra $\mathbb{C}[[T_e\mathbb{O}]]$ has the essentially unique quantization, the completed Weyl algebra, the completion at 0 of:

$$\mathbb{A}_\hbar = T(V)/(u\otimes v - v\otimes u - \hbar^2\omega(u,v))$$

Then we can argue as in the proof of Proposition in Sec. 3.1 that there is a $\mathbb{C}^\times$-equivariant $\mathbb{C}[[\hbar]]$-algebra embedding

$$\mathbb{A}_\hbar^{\wedge 0} \hookrightarrow \mathcal{U}_\hbar(\mathfrak{g})^{\wedge e}$$

Let $\mathcal{W}_\hbar'$ denote the centralizer of the image. Then (2) lifts to a $\mathbb{C}^\times$-equivariant decomposition:

$$\mathbb{A}_\hbar^{\wedge 0}\,\hat{\otimes}_{\mathbb{C}[[\hbar]]}\,\mathcal{W}_\hbar' \xrightarrow{\sim} \mathcal{U}_\hbar(\mathfrak{g})^{\wedge e}$$

We get $\mathcal{W}_\hbar'/(\hbar) \xrightarrow{\sim} \mathbb{C}[S]^{\wedge e}$. Set $\mathcal{W}_\hbar = (\mathcal{W}_\hbar')_{\mathbb{C}^\times\text{-fin}}$. This is a positively graded $\mathbb{C}[\hbar]$-algebra w. $\mathcal{W}_\hbar/(\hbar) \xrightarrow{\sim} \mathbb{C}[S]$. Finally, set $\mathcal{W} := \mathcal{W}_\hbar/(\hbar-1)$, this is a filtered quantization of $\mathbb{C}[S]$.

One can show that the two constructions give the same result, but this is nontrivial.

4) What's next?

There are functors between various categories of modules (or bimodules) over $\mathcal{U}(\mathfrak{g})$ & $\mathcal{W}$. For example, we can consider (generalized) Whittaker modules over $\mathcal{U}(\mathfrak{g})$: those where $\mathfrak{n}$ from Sec 2.1 acts with generalized eigenvalue ψ. Denote this category by Wh_e. We have $Wh_e \xrightarrow{\sim} \mathcal{W}$-mod via the $\mathcal{U}(\mathfrak{g})$-$\mathcal{W}$-bimodule $\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\{x - \langle\psi, x\rangle \mid x \in \mathfrak{n}\}$

There are more involved functors (between categories of HC bimodules over $\mathcal{U}(\mathfrak{g})$ & over $\mathcal{W}$ and between categories $\mathcal{O}$) that are much easier to construct & study using the quantum slice construction.

Then there is the affine story. A traditional general construction is via a version of Hamiltonian reduction (for the principal case one also has construction via screening operators that allows to see the Feigin-Frenkel duality).

Recently I've been thinking about a quantum slice construction in the affine setup and I'm 90% sure it can be done.